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# Group-Characters of Various Linear Groups

## A DISSERTATION

SUBMITTED TO THE FACULTY OF THE OGDEN GRADUATE SCHOOL OF SCIENCE  
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DEPARTMENT OF MATHEMATICS

BY

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## ***Group-Characters of Various Types of Linear Groups.\****

BY HERBERT E. JORDAN.

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### INTRODUCTION.

In an article entitled *Über Gruppencharaktere*† Frobenius has determined the group-characters of the group of all binary linear fractional substitutions of determinant unity (when in their normal forms), the coefficients being taken modulo  $p$ , an odd prime. In the present paper the same method is applied to more general types of groups.

In Part I we consider the group  $H \equiv SLH(2, p^n)$ ,  $p > 2$ , of all binary linear homogeneous substitutions in the  $GF[p^n]$ , of determinant unity. By the aid of two theorems due to Frobenius on the relation between the characters of a group and those of one of its quotient-groups, we deduce as a corollary the characters of the group  $F \equiv LF(2, p^n)$ ,  $p > 2$ , of all binary linear fractional substitutions in the  $GF[p^n]$  of determinant unity (when in their normal forms). We have also obtained these characters directly by the method applied to the group  $H$ ; the chief points of difference in the treatment are stated in foot-notes. The results are a direct generalization of those obtained by Frobenius. In Part II we consider the group  $H_1 \equiv SLH(2, p^n)$ ,  $p = 2$ . This is identical with the group  $LF(2, p^n)$ ,  $p = 2$ . Part III deals with the group  $F_1$  of all binary linear fractional substitutions in the  $GF[p^n]$ ,  $p > 2$ , of determinant not zero. The group  $H$  is treated with considerable detail; the others briefly.

Frobenius‡ has determined by another method the group-characters of the

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\* The abstract of the above paper appeared in the *Bulletin of the American Mathematical Society*, April, 1904. Just recently Schur has computed by different methods the characters of these same types of groups: *Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen*, Crelle, Vol. 132, 1906-7, (Heft 2).

† *Berliner Sitzungsberichte*, 1896, pp. 985-1021.

‡ *Ueber die Composition der Charaktere einer Gruppe*, *Berliner Sitzungsberichte*, 1899, pp. 880-889.

groups  $SLH(2, 3)$ ,  $(2, 5)$ , and of the alternating group on six letters of order 360, which is isomorphic with the group  $LF(2, 3^2)$  of determinant unity. Burnside\* has obtained the group-characters of the binary linear homogeneous group in the  $GF[2^8]$  of order 504. The results in this paper agree with those for the above special groups.

## I.

*The Binary Linear Homogeneous Group  $H$  in the  $GF[p^n]$ ,  $p > 2$ , of Determinant Unity.*

## § 1.

The order of the group  $H$  is  $h = p^n(p^n - 1)$ .† For the substitution

$$R: \quad \begin{aligned} x' &= \alpha x + \beta y, \\ y' &= \gamma x + \delta y, \end{aligned} \quad \alpha\delta - \beta\gamma = 1,$$

we use the notation‡  $R = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ .

We first reduce the substitutions of  $H$  to their canonical forms.§ For this purpose we consider the characteristic equation

$$K^2 - K(\alpha + \delta) + 1 = 0 \tag{1}$$

of the substitution  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ . If the roots of this equation are distinct we get the canonical form  $A$  or  $B$ :

$$A: \quad \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}, \quad \rho \text{ a mark } \neq 0 \text{ of the } GF[p^n],$$

$$B: \quad \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}, \quad \sigma \text{ a root of } \sigma^{s+1} = 1,$$

according as the equation (1) is reducible or irreducible in the  $GF[p^n]$ . If the

\* Proc. Lond. Math. Soc., Series 2, Vol. I—Part 2, p. 116.

† We shall throughout denote  $p^n$  by  $s$ , except in the notation  $GF[p^n]$ .

‡ For the substitution  $R$  taken fractionally we use the notation  $R = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ . To the two substitutions  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  and  $\begin{pmatrix} -\alpha & -\beta \\ -\gamma & -\delta \end{pmatrix}$  of  $H$  corresponds the one substitution  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  of  $F$ . We have therefore a two-to-one correspondence between  $H$  and  $F$ .

§ Dickson Linear Groups, §§ 214-216, 225.

roots of (1) are equal they must be  $+1$  or  $-1$ . We obtain then (possibly by a transformation of determinant not unity) the canonical form

$$C: \begin{pmatrix} \pm 1, & \pm 1 \\ 0, & \pm 1 \end{pmatrix}.$$

We define  $\kappa = \frac{1}{2}(\alpha + \delta)$  as the invariant\* of the substitution  $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$ .

If two substitutions of  $H$  have the same invariant, they have the same characteristic equation, and therefore the same canonical form. If two substitutions  $U$  and  $V$  of  $H$  have the same canonical form, there exists† a binary linear homogeneous substitution  $W$  belonging to the  $GF[p^n]$  (but not necessarily of determinant unity) such that  $U = W^{-1}VW$ . Precisely as in §225 (Dickson, *Linear Groups*) we can prove that if  $U$  and  $V$  have the same canonical form  $A$  or  $B$ , there exists a substitution  $W_1$  of  $H$  which transforms  $U$  into  $V$ ; also that every substitution of  $H$  of invariant  $\pm 1$  (except  $\begin{pmatrix} -1, & 0 \\ 0, & -1 \end{pmatrix}$  and the identity) is conjugate within  $H$  to one or other of the types

$$C_0: \begin{pmatrix} \pm 1, & \pm 1 \\ 0, & \pm 1 \end{pmatrix},$$

$$C_1: \begin{pmatrix} \pm 1, & \pm \mu \\ 0, & \pm 1 \end{pmatrix},$$

where  $\mu$  is a particular not-square in the  $GF[p^n]$ ; and further that the two types  $C_0, C_1$  are not conjugate within  $H$ . Hence we have the result:

*Two substitutions of  $H$  having the same invariant (not  $\pm 1$ ) are conjugate within  $H$ .*

A) Let  $\rho$  be a primitive root of the  $GF[p^n]$ . The substitution  $R = \begin{pmatrix} \rho, & 0 \\ 0, & \rho^{-1} \end{pmatrix}$  is of period  $s - 1$ . To study the conjugacy of the substitutions  $R^a = \begin{pmatrix} \rho^a, & 0 \\ 0, & \rho^{-a} \end{pmatrix}$  we transform  $R^a$  by  $U = \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$ ,  $\alpha\delta - \beta\gamma = 1$ , and obtain

$$V = U^{-1}R^aU = \begin{pmatrix} \alpha\delta\rho^a - \beta\gamma\rho^{-a}, & -\alpha\beta(\rho^a - \rho^{-a}) \\ \gamma\delta(\rho^a - \rho^{-a}), & -\beta\gamma\rho^a + \alpha\delta\rho^{-a} \end{pmatrix}.$$

In order that  $V$  shall be identical with  $R^a$  (i. e.,  $U$  commutative with  $R^a$ ) it is

\* In the case of  $F$  we define  $\kappa = \pm \frac{1}{2}(\alpha + \delta)$  as the invariant of the substitution  $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$ .

† Dickson, *Linear Groups*, § 216.

necessary that either  $\alpha\beta = \gamma\delta = 0$ , or  $\rho^a = \rho^{-a} = 0$ . The first alternative leads to two cases:

- 1) if  $\beta = \gamma = 0$  then  $V = \begin{pmatrix} \rho^a & 0 \\ 0 & \rho^{-a} \end{pmatrix} = R^a$ ,  $U = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ ;
- 2) if  $\alpha = \delta = 0$  then  $V = \begin{pmatrix} \rho^{-a} & 0 \\ 0 & \rho^a \end{pmatrix} = R^{-a}$ ,  $U = \begin{pmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{pmatrix}$ .

If  $R^a \neq R^{-a}$  we have  $s-1$  substitutions  $U$  commutative with  $R^a$ ; and therefore  $R^a$  is one of  $s(s^2-1) \div (s-1) = s(s+1)$  conjugate substitutions. If  $R^a = R^{-a}$  then  $\rho^a = \rho^{-a}$ , which is the second alternative. According as  $\rho^a = +1$  or  $-1$ ,  $R^a$  is the identity or  $R^{\frac{s-1}{2}} \equiv \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ ; each of these substitutions is conjugate only to itself. With the exception of these the powers of  $R$  are conjugate in pairs, thus representing  $\frac{1}{2}(s-3)$  classes of conjugate substitutions, each class containing  $s(s+1)$  substitutions.\*

B) The group  $H$  is holodrically isomorphic with the group†  $G_{2,n}$  of substitutions

$$U = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}, \quad (A\bar{A} + B\bar{B} = 1),$$

where  $\bar{A} \equiv A^*$  is the conjugate of  $A$  with respect to the  $GF[p^*]$ . If  $\sigma$  is a primitive root of the equation  $\sigma^{s+1} = 1$ , so that  $\bar{\sigma} = \sigma^{-1}$ , then the substitution  $S = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$  is of period  $s+1$ . As above we find that the powers of  $S$ , except  $S^{\frac{s+1}{2}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$  and  $S^{s+1}$ , which is the identity, are conjugate in pairs, viz.,  $S^b$  with  $S^{-b}$ . There are  $\frac{1}{2}(s-1)$  classes represented by the powers of  $S$ , each containing  $s(s-1)$  substitutions.‡

\*In the case of the group  $F$  if  $R^a = R^{-a}$  either  $\rho^a = \rho^{-a}$ , i. e.,  $R^a$  is the identity, or  $\rho^a = -\rho^{-a}$ , which is possible only if  $s-1$  is divisible by 4;  $R^a \equiv R^{\frac{s-1}{2}}$  is commutative with  $s-1$  substitutions  $U$ , and is therefore one of  $\frac{1}{2}s(s+1)$  conjugate substitutions. If we define  $\epsilon$  as  $+1$  or  $-1$  according as  $s$  has the form  $4l+1$  or  $4l-1$ , where  $l$  is an integer, then we have  $\frac{1}{2}(s-2+\epsilon)$  classes represented by the powers of  $R$ , each containing  $(s+1)$  substitutions except the class of period two, which contains  $\frac{1}{2}s(s+1)$  substitutions.

†Dickson, *Linear Groups*, p. 132.

‡The substitution  $S = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$  of  $F$  is of period  $\frac{1}{2}(s+1)$ ; the substitutions  $S^b$  (not the identity) are conjugate in pairs except when  $\epsilon = -1$ , and then  $S^{\frac{s+1}{2}}$  is conjugate only to itself and is of period two. We have  $\frac{1}{2}(s-\epsilon)$  classes, each containing  $s(s-1)$  substitutions, except the class of period two, which contains  $\frac{1}{2}s(s-1)$  substitutions.

The numbers  $\pm a$  ( $\pm b$ ) taken mod.  $s - 1$  (mod.  $s + 1$ ) will be called indifferently the index of the class represented by  $R^a(S^b)$ . We have defined  $\kappa = \frac{1}{2}(\alpha + \delta)$  as the invariant of the substitution  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ . The substitutions\*  $R^a(S^b)$  are characterized by the property that  $\kappa^2 - 1$  is a square (not-square) in the  $GF[p^n]$ .

C) Consider the substitution

$$T_\mu = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \mu \text{ a mark } \neq 0 \text{ of the } GF[p^n].$$

Transforming  $T_\mu$  by  $U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ ,  $\alpha\delta - \beta\gamma = 1$ , we obtain

$$V = U^{-1} T_\mu U = \begin{pmatrix} 1 - \alpha\gamma\mu & \alpha^2\mu \\ -\gamma^2\mu & 1 + \alpha\gamma\mu \end{pmatrix}.$$

$T_\mu$  is commutative only with those substitutions  $U$  for which  $\gamma = 0$ ,  $\alpha = \pm 1$ , in number  $2s$ ; hence  $T_\mu$  is one of  $s(s^2 - 1) \div 2s = \frac{1}{2}(s^2 - 1)$  conjugate substitutions. We observe that the conjugate substitutions  $T_\mu$  and  $V$  have the property that  $\mu$  and  $\alpha^2\mu$  ( $\alpha \neq 0$ ), or  $\mu$  and  $\gamma^2\mu$  in case  $\alpha = 0$ , are both squares or both not-squares in the  $GF[p^n]$ . This condition can easily be proved to be sufficient for the conjugacy of  $T_\mu$  and  $V$ ; i. e., a substitution  $Q = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$ ,  $\alpha'\delta' - \beta'\gamma' = 1$ , of invariant unity is conjugate to  $T_\mu$  if  $\mu$  and  $\beta'$  ( $\neq 0$ ), or  $\mu$  and  $-\gamma'$  in case  $\beta' = 0$ , are both squares or both not-squares in the  $GF[p^n]$ .

The substitutions  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  (except the identity) of invariant  $+1$  will be said to belong to the class  $(\mu)$  or to the class  $(\nu)$  according as  $\beta$  ( $\neq 0$ ), or  $-\gamma$  if  $\beta = 0$ , is a square or a not-square in the  $GF[p^n]$ . Similarly the substitutions  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ , except  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ , of invariant  $-1$  will be said to belong to the class†  $(m)$  or to the class  $(n)$  according as  $\beta$  ( $\neq 0$ ), or  $-\gamma$  if  $\beta = 0$ , is a square or a not-square in the  $GF[p^n]$ .

\* The substitutions of period two have the invariant zero.

† To the two classes  $(\mu)$  and  $(m)$  of  $H$  corresponds the one class  $(\mu)$  of  $F$ ; and similarly for  $(\nu)$  and  $(n)$ .

The substitutions\*  $\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$  and  $\begin{pmatrix} -1, & 0 \\ 0, & -1 \end{pmatrix}$  will be denoted by  $(\lambda)$  and  $(I)$  respectively.

The total number† of classes of conjugate substitutions is  $\frac{1}{2}(s-3) + \frac{1}{2}(s-1) + 4 + 2 = s + 4$ .

Since the powers of  $R$  and  $S$  were shown to be conjugate with their reciprocals, a substitution  $W$  and its reciprocal  $W^{-1}$  belong to the same class except when  $W$  belongs to one of the classes  $(\mu)$ ,  $(\nu)$ ,  $(m)$ ,  $(n)$ .

## § 2.

We define  $\varepsilon_x$  to be  $+1$  or  $-1$  according as  $x^2 - 1$  is a square or a not-square in the  $GF[p^n]$ , where  $x$  denotes henceforth any invariant except  $\pm 1$ . In place of  $\varepsilon_0$  we write  $\varepsilon.\ddagger$

The class whose invariant is  $x$  is denoted by  $(x)$ . This notation is unique. For, two conjugate substitutions have the same invariant; and it has been proved that if two substitutions of  $H$  have the same invariant (not  $\pm 1$ ) they are conjugate. Instead of  $x$  we shall nearly always use  $\alpha, \beta, \gamma, \dots$ , and we shall denote the indices of the classes  $(\alpha), (\beta), (\gamma), \dots$  by  $\pm a, \pm b, \pm c, \dots$  respectively. These indices are taken mod.  $s-1$  or mod.  $s+1$  according as  $x^2 - 1$  is a square or a not-square in the  $GF[p^n]$ .

Denoting by  $h_\theta$  the number of substitutions in a class  $(\theta)$  we have§

$$h_\lambda = h_i = 1, \quad h_\mu = h_\nu = h_m = h_n = \frac{1}{2}(s^2 - 1), \quad h_x = s(s + \varepsilon_x).$$

If  $\varepsilon_\alpha = \varepsilon$ ,  $\varepsilon_\beta = -\varepsilon$ , the numbers|| of classes  $(\alpha)$  and  $(\beta)$  are  $\frac{1}{2}(s-2-\varepsilon)$  and  $\frac{1}{2}(s-2+\varepsilon)$  respectively.

We define¶  $\zeta_x$  as  $+1$  or  $-1$  according as  $-2(1-x)$  is a square or a not-square in the  $GF[p^n]$ . Then  $\zeta_\alpha = \varepsilon_\alpha(-1)^a$ .

\* The one corresponding substitution  $\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$  of  $F$  will be denoted by  $(\lambda)$ .

† For  $F$  the total number of classes is  $\frac{1}{2}(s+5)$ .

‡ For the definition of  $\varepsilon$  compare p. 390, foot-note\*.

§ For  $F$  we have  $h_\lambda = 1$ ,  $h_\mu = h_\nu = \frac{1}{2}(s^2 - 1)$ ,  $h_x = s(s + \varepsilon_x)$ ,  $h_0 = \frac{1}{2}s(s + \varepsilon)$ . The class  $(0)$  requires to be distinguished from the other classes  $(\kappa)$  more frequently in the case of  $F$  than in the case of  $H$ .

|| For  $F$  the numbers of classes  $(\alpha)$  and  $(\beta)$  are  $\frac{1}{2}(s-\varepsilon)$  and  $\frac{1}{2}(s-2+\varepsilon)$  respectively.

¶ In the case of  $F$  we define  $\eta_\kappa$  as  $+1, -1$ , or  $0$  according as  $-2(1+\kappa)$  and  $-2(1-\kappa)$  are both squares, both not-squares, or one a square and the other a not-square, in the  $GF[p^n]$ . We also define  $2\eta_\kappa = \varepsilon$ . If  $\varepsilon_\alpha = \varepsilon$  then  $\eta_\alpha = (-1)^a \varepsilon$ ; if  $\varepsilon_\beta = -\varepsilon$  then  $\eta_\beta = 0$ ; further,  $\eta_\kappa = \frac{1}{2}(1 + \varepsilon_\kappa) \zeta_\kappa$ .

## § 3.

Three (distinct or equal) classes  $(\alpha)$ ,  $(\beta)$ ,  $(\gamma)$  are called *concordant* if between their invariants there exists the relation \*

$$\alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta\gamma = 1; \quad (1)$$

otherwise they are called *discordant*. If we write (1) in the form

$$(\alpha^2 - 1)(\beta^2 - 1) = (\alpha\beta - \gamma)^2$$

it follows that  $\varepsilon_\alpha = \varepsilon_\beta$ ; similarly  $\varepsilon_\beta = \varepsilon_\gamma$ . Hence three concordant classes must all be represented by powers of  $R$  or all by powers of  $S$ . If  $\alpha$  and  $\beta$  are given we find that the classes whose invariants are

$$\gamma = \alpha\beta + \sqrt{(\alpha^2 - 1)(\beta^2 - 1)}, \quad \delta = \alpha\beta - \sqrt{(\alpha^2 - 1)(\beta^2 - 1)} \quad (2)$$

are concordant with  $(\alpha)$  and  $(\beta)$ . If  $\beta = \alpha$  then  $\gamma = 2\alpha^2 - 1$ ; and therefore we have  $\varepsilon_\alpha = \varepsilon_{2\alpha^2 - 1} - 1$ .

Let  $r$  denote  $\rho$  or  $\sigma$  according as  $\varepsilon_\alpha = +1$  or  $-1$ . Substituting the values  $2\alpha = r^a + r^{-a}$ , etc., in (1), and factoring, we obtain

$$(r^{a+b+c} - 1)(r^{-a-b+c} - 1)(r^{-a+b-c} - 1)(r^{a-b-c} - 1) = 0.$$

Hence  $a \pm b \pm c \equiv 0 \pmod{s-1}$  or  $s+1$  according as  $\varepsilon_\alpha = +1$  or  $-1$ . The indices of the two classes  $(\gamma)$  and  $(\delta)$  concordant with  $(\alpha)$  and  $(\beta)$  are therefore

$$c \equiv a + b, \quad d \equiv a - b \pmod{s-1, s+1 \text{ respectively}}.$$

## § 4.

Let  $\Theta$ ,  $\Phi$ ,  $\Psi$  represent the substitutions of any three distinct or equal classes  $(\theta)$ ,  $(\phi)$ ,  $(\psi)$  respectively, and let  $(\theta')$ ,  $(\phi')$ ,  $(\psi')$  denote the classes of the inverse substitutions  $\Theta^{-1}$ ,  $\Phi^{-1}$ ,  $\Psi^{-1}$  respectively. If  $\Theta$ ,  $\Phi$ ,  $\Psi$  run through all the substitutions of their respective classes, we denote† by  $h_{\theta\phi\psi}$  the number of times we obtain the relation  $\Theta\Phi\Psi = E$  (the identity), or  $\Theta\Phi = \Psi^{-1}$ . The subscripts  $\theta$ ,  $\phi$ ,  $\psi$  may be permuted in any manner.

To obtain  $h_{\alpha\beta\gamma}$  we determine  $\frac{h_{\alpha\beta\gamma}}{h_\alpha}$ ; we take a *particular* substitution of  $(\alpha)$ , compound it with all the substitutions of  $(\beta)$ ,

$$\begin{pmatrix} \alpha & 1 \\ \alpha^2 - 1 & \alpha \end{pmatrix} \begin{pmatrix} \xi & \eta \\ \zeta & 2\beta - \xi \end{pmatrix} = \begin{pmatrix} \alpha\xi + \eta(\alpha^2 - 1) & \xi + \alpha\eta \\ \alpha\zeta + (2\beta - \xi)(\alpha^2 - 1) & \zeta + \alpha(2\beta - \xi) \end{pmatrix},$$

and determine how many of the resulting substitutions belong to the class  $(\gamma')$ .

\* For  $F$  this relation takes the form  $\alpha^2 + \beta^2 + \gamma^2 \pm 2\alpha\beta\gamma = 1$ .

† Frobenius, *Über Gruppencharaktere*, 1896, pp. 987, 988.

In order that the resulting substitutions may be of class  $(\gamma') \equiv (\gamma)^*$  and have the determinant unity, we must have

$$\begin{aligned} \alpha\xi + \eta(\alpha^2 - 1) + \zeta + \alpha(2\beta - \xi) &= 2\gamma, \\ \xi(2\gamma - \xi) - \eta\zeta &= 1. \end{aligned}$$

The number of distinct sets of solutions  $\xi, \eta, \zeta$  of these equations will give  $\frac{h_{\alpha\beta\gamma}}{h_\alpha}$ . Eliminating  $\zeta$  we obtain

$$(\xi - \beta)^2 - (\alpha^2 - 1)\left(\eta + \frac{\alpha\beta - \gamma}{\alpha^2 - 1}\right)^2 = \frac{(\alpha^2 - 1)(\beta^2 - 1) - (\alpha\beta - \gamma)^2}{\alpha^2 - 1}.$$

If  $(\alpha), (\beta), (\gamma)$  are discordant the right-hand side is distinct from zero, and we obtain  $s - \epsilon_\alpha$  sets of solutions†. If  $(\alpha), (\beta), (\gamma)$  are concordant‡ the right-hand side is zero, and we obtain  $s + \epsilon_\alpha(s - 1)$  sets of solutions. Hence  $h_{\alpha\beta\gamma} = h + \epsilon_\alpha s^2(s + \epsilon_\alpha)$  or  $h$  according as  $(\alpha), (\beta), (\gamma)$  are concordant or discordant.

If we denote the substitutions of  $(\mu)$  and  $(\nu)$  by  $P$  and  $Q$  respectively, then according as  $s = 1$  or  $-1$  will  $P^{-1}$  belong to  $(\mu)$  or  $(\nu)$ , and  $Q^{-1}$  to  $(\nu)$  or  $(\mu)$ . Hence  $h_{\lambda\mu\nu} = \frac{1}{2}h_\mu(1 - \epsilon)$ . Similarly  $h_{\lambda\mu\alpha} = \frac{1}{2}h_\mu(1 - \epsilon)$ .

The group  $H$  is self-conjugate under the group of all binary linear homogeneous substitutions of determinant  $\neq 0$ ; by a substitution of determinant a not-square in the  $GF[p^n]$  the class  $(\mu)$  is transformed into  $(\nu)$ , and  $(\nu)$  into  $(\mu)$ , and simultaneously  $(m)$  into  $(n)$  and  $(n)$  into  $(m)$ . Hence the notations  $(\mu)$  and  $(\nu)$  are interchangeable; likewise  $(m)$  and  $(n)$ ; furthermore the interchange of  $(\mu)$  and  $(\nu)$  must be accompanied by the interchange of  $(m)$  and  $(n)$ , and vice versa.

To determine  $h_{\mu\nu}$  we compute  $h_\mu(h_{\mu\nu\mu} + h_{\mu\nu\nu} + h_{\mu\nu\lambda})$ ; we take a definite substitution of  $(\mu)$ , compound it with all the substitutions of  $(\nu)$ ,

$$\begin{pmatrix} 0, 1 \\ -1, 2 \end{pmatrix} \begin{pmatrix} \xi, \eta \\ \zeta, 2 - \xi \end{pmatrix} = \begin{pmatrix} -\eta, & \xi + 2\eta \\ -2 + \xi, & \zeta + 4 - 2\xi \end{pmatrix},$$

\* See last paragraph of §1.

† Dickson, *Linear Groups*, p. 46.

‡ In the case of  $F$  we have the equations

$$(\xi - \beta)^2 - (\alpha^2 - 1)\left(\eta + \frac{\alpha\beta \pm \gamma}{\alpha^2 - 1}\right)^2 = \frac{(\alpha^2 - 1)(\beta^2 - 1) - (\alpha\beta \pm \gamma)^2}{\alpha^2 - 1}.$$

If  $(\alpha), (\beta), (\gamma)$  are concordant then one of the relations  $(\alpha^2 - 1)(\beta^2 - 1) - (\alpha\beta \pm \gamma)^2 = 0$  holds and not the other. Suppose that  $(\alpha^2 - 1)(\beta^2 - 1) - (\alpha\beta + \gamma)^2 = 0$ ; then the equations with the upper and lower signs have  $s + \epsilon_\alpha(s - 1)$  and  $s - \epsilon_\alpha$  sets of solutions respectively; in all  $2(s - \epsilon_\alpha) + \epsilon_\alpha s$  sets of solutions.

and find how many of the resulting substitutions belong to the classes  $(\lambda)$ ,  $(\mu)$ ,  $(\nu)$  collectively, i. e., have the invariant  $+1$ . We have therefore the equations

$$\begin{aligned}\xi(2 - \xi) - \eta\zeta &= 1, \\ -\eta + \zeta + 4 - 2\xi &= 2.\end{aligned}$$

Eliminating  $\xi$  we get  $(\eta + \zeta)^2 = 0$ , or  $-\zeta = \eta$ . We can take for  $\eta$  every not-square in the  $GF[p^n]$ , thus obtaining  $\frac{1}{2}(s-1)$  sets of solutions. But  $h_{\mu\nu\mu} = h_{\nu\mu\nu} = h_{\mu\nu\nu}$ ; hence  $h_{\mu\mu\nu} = \frac{1}{2}h_\mu(s-2+\epsilon)$ .

Proceeding in this way we get the following results.\*

$$\begin{aligned}h_{\alpha\beta\gamma} &= h + \epsilon_\alpha s^2(s + \epsilon_\alpha) \text{ or } h \text{ according as } (\alpha), (\beta), (\gamma) \text{ are concordant or discordant,} \\ h_{\alpha\alpha\lambda} &= h_\alpha, h_{\alpha\alpha l} = 0, h_{\alpha\alpha\mu} = h_{\alpha\alpha\nu} = \frac{1}{2}h(1 + \epsilon_\alpha), h_{\alpha\alpha m} = h_{\alpha\alpha n} = \frac{1}{2}h \text{ if } \alpha \neq 0, \\ h_{\alpha\alpha m} &= h_{\alpha\alpha n} = \frac{1}{2}h(1 + \epsilon), h_{-\alpha\alpha l} = h_\alpha, h_{-\alpha\alpha m} = \frac{1}{2}h(1 + \epsilon_\alpha), \\ h_{\alpha\beta\lambda} &= 0, h_{\alpha\beta\mu} = h_{\alpha\beta\nu} = h_{\alpha\beta m} = h_{\alpha\beta n} = \frac{1}{2}h, \\ h_{\alpha\lambda l} &= h_{\alpha\lambda\mu} = h_{\alpha\lambda\nu} = h_{\alpha\lambda m} = h_{\alpha\lambda n} = h_{\alpha l\mu} = h_{\alpha l\nu} = h_{\alpha l m} = h_{\alpha l n} = 0, \\ h_{\alpha\mu\mu} &= h_{\alpha\nu\nu} = h_{\alpha m m} = h_{\alpha n n} = \frac{1}{2}h(1 + \epsilon\zeta_\alpha), h_{\alpha\mu\nu} = h_{\alpha m n} = \frac{1}{2}h(1 - \epsilon\zeta_\alpha), \\ h_{\alpha\mu m} &= h_{\alpha\nu n} = \frac{1}{2}h(1 + \epsilon_\alpha\zeta_\alpha), h_{\alpha m\nu} = h_{\alpha n\mu} = \frac{1}{2}h(1 - \epsilon_\alpha\zeta_\alpha), \\ h_{l\mu m} &= h_{l\nu n} = h_{\lambda\mu\mu} = h_{\lambda\nu\nu} = h_{\lambda m m} = h_{\lambda n n} = \frac{1}{2}h_\mu(1 + \epsilon), \\ h_{\lambda m\nu} &= h_{\lambda\mu n} = h_{\lambda\mu m} = h_{\lambda\nu n} = h_{l\mu\mu} = h_{l\nu\nu} = h_{l\mu\nu} = h_{l m n} = 0, \\ h_{l m m} &= h_{l n n} = h_{\mu\mu n} = h_{m\nu\nu} = h_{\mu m\nu} = h_{\mu\nu n} = 0, \\ h_{m m n} &= h_{m n n} = 0, h_{\lambda\mu\nu} = h_{\lambda m n} = h_{l\mu n} = h_{l m\nu} = \frac{1}{2}h_\mu(1 - \epsilon), \\ h_{\mu\mu\mu} &= h_{\nu\nu\nu} = h_{\mu m m} = h_{\nu n n} = \frac{1}{2}h_\mu(s - 2 - 3\epsilon), \\ h_{\mu\mu\nu} &= h_{\mu\nu\nu} = h_{\mu n n} = h_{m m\nu} = h_{\mu m n} = h_{m\nu n} = \frac{1}{2}h_\mu(s - 2 + \epsilon), \\ h_{\mu\mu n} &= h_{\nu\nu n} = h_{m m m} = h_{n n n} = s h_\mu.\end{aligned}$$

## § 5.

The value of a group-character  $\chi$  for any class  $(\theta)$  is denoted by  $\chi_\theta$ ; but sometimes the value of a group-character for a class represented by a power of  $R$  or  $S$  is denoted by  $\chi(R^a)$  or  $\chi(S^b)$  respectively. Also  $\chi_\lambda = f$ .

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\* These results reduce to a very few in the case of  $\mathcal{H}$ , since  $(l) = (\lambda)$ ,  $(m) = (\mu)$ ,  $(n) = (\nu)$ . For  $\mathcal{H}$  we have the following additional results:  $h_{\alpha\alpha\beta} = 2h + \epsilon s^2(s + \epsilon)$  or  $2h$  according as  $(\alpha)$ ,  $(\alpha)$ ,  $(\beta)$  are concordant or discordant;  $h_{\alpha\alpha\alpha} = \frac{1}{2}h$ ,  $h_{\alpha\alpha\alpha} = h$ ,  $h_{\alpha\alpha l} = \frac{1}{2}s(s + \epsilon)$ ,  $h_{\alpha\alpha\mu} = h_{\alpha\alpha\nu} = \frac{1}{2}h(1 + \epsilon)$ ,  $h_{\alpha\alpha\mu} = h_{\alpha\alpha\nu} = h$ ,  $h_{\alpha\mu\mu} = h_{\alpha\nu\nu} = \frac{1}{2}h(1 + \epsilon\eta)$ ,  $h_{\alpha\mu\nu} = \frac{1}{2}h(1 - \epsilon\eta)$ , where  $\eta = \eta_\alpha$ .

We make the following abbreviations:\*

$$x = \frac{1}{2}(\chi_\mu + \chi_\nu + \chi_m + \chi_n) + \sum_\kappa \chi_\kappa,$$

$$y = \frac{1}{2}(\chi_\mu + \chi_\nu + \chi_m + \chi_n) + \sum_\kappa \zeta_\kappa \chi_\kappa,$$

$$z = \frac{1}{2}(\chi_\mu + \chi_\nu + \chi_m + \chi_n) + \sum_\kappa \varepsilon_\kappa \zeta_\kappa \chi_\kappa.$$

From the relation †

$$h_\theta h_\psi \chi_\phi \chi_\psi = f \sum_\phi h_{\theta\psi\phi} \chi_\phi,$$

where  $(\theta)$ ,  $(\phi)$ ,  $(\psi)$  are any three classes, we derive the following set of equations ‡

$$\chi_i^2 = f^2, \quad (1)$$

$$\chi_\alpha \chi_i = f \chi_{-\alpha}, \quad (2)$$

$$s \chi_\alpha \chi_\beta = f x, \quad (\varepsilon_\beta = -\varepsilon_\alpha), \quad (3)$$

$$\frac{s(s+\varepsilon_\alpha)}{f} \chi_\alpha \chi_\beta = x(s-\varepsilon_\alpha) + \varepsilon_\alpha s(\chi_\gamma + \chi_\delta), \quad (\varepsilon_\beta = \varepsilon_\alpha, \alpha \neq -\beta), \quad (4)$$

where  $\gamma$  and  $\delta$  are determined by (3), §3,

$$\frac{s(s+\varepsilon_\alpha)}{f} \chi_\alpha \chi_{-\alpha} = \chi_i + x(s-\varepsilon_\alpha) + \frac{1}{2} \varepsilon_\alpha (s-\varepsilon_\alpha)(\chi_m + \chi_n) + \varepsilon_\alpha s \chi_{2\alpha^2-1}, \quad (5)$$

$$\frac{s(s+\varepsilon_\alpha)}{f} \chi_\alpha^2 = f + x(s-\varepsilon_\alpha) + \frac{1}{2} \varepsilon_\alpha (s-\varepsilon_\alpha)(\chi_\mu + \chi_\nu) + \varepsilon_\alpha s \chi_{2\alpha^2-1}, \quad (6)$$

$$\frac{s(s+\varepsilon)}{f} \chi_\delta^2 = f + x(s-\varepsilon) + \frac{1}{2} \varepsilon (s-\varepsilon)(\chi_\mu + \chi_\nu + \chi_m + \chi_n), \quad (7)$$

$$\frac{s+\varepsilon_\alpha}{f} \chi_\alpha \chi_\mu = x + \varepsilon_\alpha \chi_\alpha + \frac{1}{2} \zeta_\alpha (\chi_\mu - \chi_\nu) + \frac{1}{2} \varepsilon \varepsilon_\alpha \zeta_\alpha (\chi_m - \chi_n), \quad (8)$$

$$\frac{s+\varepsilon_\alpha}{f} \chi_\alpha \chi_\nu = x + \varepsilon_\alpha \chi_\alpha - \frac{1}{2} \zeta_\alpha (\chi_\mu - \chi_\nu) - \frac{1}{2} \varepsilon \varepsilon_\alpha \zeta_\alpha (\chi_m - \chi_n), \quad (8a)$$

$$\frac{s+\varepsilon_\alpha}{f} \chi_\alpha \chi_m = x + \varepsilon_\alpha \chi_\alpha + \frac{1}{2} \varepsilon \varepsilon_\alpha \zeta_\alpha (\chi_\mu - \chi_\nu) + \frac{1}{2} \zeta_\alpha (\chi_m - \chi_n), \quad (9)$$

$$\frac{s+\varepsilon_\alpha}{f} \chi_\alpha \chi_n = x + \varepsilon_\alpha \chi_\alpha - \frac{1}{2} \varepsilon \varepsilon_\alpha \zeta_\alpha (\chi_\mu - \chi_\nu) - \frac{1}{2} \zeta_\alpha (\chi_m - \chi_n), \quad (9a)$$

\*In the case of  $F'$  we make the abbreviations

$$x = \chi_0 + \chi_\mu + \chi_\nu + 2 \sum_\kappa \chi_\kappa, \quad \kappa \neq 0,$$

$$y = \eta_0 \chi_0 + \eta_1 (\chi_\mu + \chi_\nu) + 2 \sum_\kappa \eta_\kappa \chi_\kappa.$$

† Über Gruppencharaktere, p. 994.

‡ We get the equations for  $F'$  from (3), (4), (6)–(8a), (10)–(10b) if we set  $\chi_m = \chi_\mu$ ,  $\chi_n = \chi_\nu$ , and remember that  $\frac{1}{2}(1 + \varepsilon \varepsilon_\alpha) \zeta_\alpha = \eta_\alpha$ .

$$\begin{aligned} \frac{s^2-1}{fs} \chi_\mu^2 &= x + \varepsilon y + (1+\varepsilon) \frac{f}{s} - \frac{s+2\varepsilon+4}{4s} (\chi_\mu + \chi_\nu) \\ &\quad - \frac{1}{s} (\chi_\mu - \chi_\nu) + \frac{1}{4} (\chi_m + \chi_n) + \varepsilon (\chi_m - \chi_n), \end{aligned} \quad (10)$$

$$\begin{aligned} \frac{s^2-1}{fs} \chi_\nu^2 &= x + \varepsilon y + (1+\varepsilon) \frac{f}{s} - \frac{s+2\varepsilon+4}{4s} (\chi_\mu + \chi_\nu) + \frac{1}{s} (\chi_\mu - \chi_\nu) \\ &\quad + \frac{1}{4} (\chi_m + \chi_n) - \varepsilon (\chi_m - \chi_n), \end{aligned} \quad (10a)$$

$$\frac{s^2-1}{fs} \chi_\mu \chi_\nu = x - \varepsilon y + (1-\varepsilon) \frac{f}{s} + \frac{s+2\varepsilon-4}{4s} (\chi_\mu + \chi_\nu) - \frac{1}{4} (\chi_m + \chi_n), \quad (10b)$$

$$\begin{aligned} \frac{s^2-1}{fs} \chi_\mu \chi_m &= x + z + \frac{1+\varepsilon}{s} \chi_i + \frac{2-\varepsilon}{4} (\chi_\mu + \chi_\nu) + \varepsilon (\chi_\mu - \chi_\nu) \\ &\quad - \frac{s\varepsilon+2\varepsilon+4}{4s} (\chi_m + \chi_n) - \frac{1}{s} (\chi_m - \chi_n), \end{aligned} \quad (11)$$

$$\begin{aligned} \frac{s^2-1}{fs} \chi_\nu \chi_n &= x + z + \frac{1+\varepsilon}{s} \chi_i + \frac{2-\varepsilon}{4} (\chi_\mu + \chi_\nu) - \varepsilon (\chi_\mu - \chi_\nu) \\ &\quad - \frac{s\varepsilon+2\varepsilon+4}{4s} (\chi_m + \chi_n) + \frac{1}{s} (\chi_m - \chi_n), \end{aligned} \quad (11a)$$

$$\begin{aligned} \frac{s^2-1}{fs} \chi_\mu \chi_n &= \frac{s^2-1}{fs} \chi_m \chi_\nu = x - z + \frac{1-\varepsilon}{s} \chi_i \\ &\quad - \frac{2-\varepsilon}{4} (\chi_\mu + \chi_\nu) + \frac{s\varepsilon+2\varepsilon-4}{4s} (\chi_m + \chi_n), \end{aligned} \quad (11b)$$

$$\chi_i \chi_\mu = f \chi_m, \quad (12)$$

$$\chi_i \chi_\nu = f \chi_n. \quad (12a)$$

I. We seek first those solutions for which  $\chi_\mu$  and  $\chi_\nu$  are distinct. Then, according to (12) and (12a),  $\chi_m$  and  $\chi_n$  are distinct. From (1) we find  $\chi_i = \pm f$ .

Suppose first that  $\chi_i = f$ . Then  $\chi_m = \chi_\mu$  and  $\chi_n = \chi_\nu$ , and from (8) and (8a) we obtain  $\frac{s+\varepsilon a}{f} \chi_n = \zeta_n (1 + \varepsilon \varepsilon_n)$ . If  $\varepsilon_n = -\varepsilon$  then  $\chi_n = 0$ ; if  $\varepsilon_n = \varepsilon$ , and if we set the proportionality factor\*  $f = \frac{1}{2}(s + \varepsilon)$ , we obtain  $\chi_n = \zeta_n$ . Hence  $\chi(R^a) = \frac{(1+\varepsilon)(-1)^a}{2}$ ,  $\chi(S^b) = -\frac{(1-\varepsilon)(-1)^b}{2}$ . According to (3),  $x = 0$ . If in (6)  $\varepsilon_n = -\varepsilon$ , then  $\chi_\mu + \chi_\nu = \varepsilon$ , and  $y = \frac{1}{2} + \sum \zeta_x^2$ , where  $\varepsilon_x = \varepsilon$ . The number of classes ( $x$ ) for which  $\varepsilon_x = \varepsilon$  is  $\frac{1}{2}(s - 2 - \varepsilon)$ , and therefore  $y = \frac{1}{2}(s - 1 - \varepsilon)$ . Similarly  $z = \frac{1}{2}(s - 2)$ . From (10b) we get  $4\chi_\mu \chi_\nu = 1 - \varepsilon s$ ;

\* The proportionality factor may be chosen arbitrarily. See Über Gruppencharaktere, p. 999.

also we already have  $\chi_\mu + \chi_\nu = \varepsilon$ . Hence  $\chi_\mu = \chi_m = \frac{1}{2}(\varepsilon \pm \sqrt{\varepsilon s})$ ,  $\chi_\nu = \chi_n = \frac{1}{2}(\varepsilon \mp \sqrt{\varepsilon s})$ . These values of the characters will be found to satisfy all the equations.

Let next  $\chi_i = -f$ . Then  $\chi_m = -\chi_\mu$ ,  $\chi_n = -\chi_\nu$ . From (8) and (8a) we get as before  $\frac{s+\varepsilon_a}{f} \chi_a = \zeta_a(1 - \varepsilon \varepsilon_a)$ . If  $\varepsilon_a = \varepsilon$ ,  $\chi_a = 0$ ; if  $\varepsilon_a = -\varepsilon$ , and if we set  $f = \frac{1}{2}(s - \varepsilon)$ , we obtain\*  $\chi_a = \zeta_a$ . According to (3),  $x = 0$ ; if in (6)  $\varepsilon_a = \varepsilon$  then  $\chi_\mu + \chi_\nu = -\varepsilon$ , and therefore  $\chi_m + \chi_n = \varepsilon$ . Also  $y = \frac{1}{2}(s - 2 + \varepsilon)$  and  $z = -\frac{1}{2}(s - 2 + \varepsilon)$ . From (10b) we obtain  $4\chi_\mu \chi_\nu = 1 - \varepsilon s$ ; this combined with  $\chi_\mu + \chi_\nu = -\varepsilon$  gives  $\chi_\mu = -\chi_m = \frac{1}{2}(-\varepsilon \pm \sqrt{\varepsilon s})$ ,  $\chi_\nu = -\chi_n = \frac{1}{2}(-\varepsilon \mp \sqrt{\varepsilon s})$ .

II. For all other solutions  $\chi_\mu = \chi_\nu$ , and therefore  $\chi_m = \chi_n$ . Instead of equations (10)–(11b) we shall use the following which are obtained from them by addition and subtraction:

$$\frac{s^2 - 1}{f} \chi_\mu^2 = s\varepsilon x + f - 2\chi_\mu, \quad (10')$$

$$2\varepsilon s y = (s + 2\varepsilon) \chi_\mu - s \chi_m - 2\varepsilon f, \quad (10'')$$

$$\frac{s^2 - 1}{f} \chi_\mu \chi_m = s\varepsilon x + \chi_i - 2\chi_m, \quad (11')$$

$$2\varepsilon s z = s(1 - 2\varepsilon) \chi_\mu + (s + 2) \chi_m - 2\chi_i. \quad (11'')$$

We seek first those solutions for which  $x$  is distinct from zero. According to (3) none of the characters  $\chi_a$  can be zero; and  $\chi_a = \chi_\nu$  if  $\varepsilon_a = \varepsilon_\nu$ , i. e., all characters  $\chi_a$  are equal for which  $\varepsilon_a$  has the same sign. Since  $\chi_a = \chi_{-a}$  it follows from (8) and (9) that  $\chi_m = \chi_\mu$ , and therefore from (12) that  $\chi_i = f$ .

Let  $\varepsilon_a = \varepsilon$ ,  $\varepsilon_\beta = -\varepsilon$ . Then\*  $y = \varepsilon + \chi_a \sum \zeta_a + \chi_\beta \sum \zeta_\beta = \varepsilon - \varepsilon \chi_a$ . From (6) and (8) we obtain

$$\chi_a = \frac{1}{s} \{(s - \varepsilon) \chi_\mu + \varepsilon f\}, \quad (A)$$

$$\chi_\beta = \frac{1}{s} \{(s + \varepsilon) \chi_\mu - \varepsilon f\}.$$

In the sum  $x = \frac{1}{2}(\chi_\mu + \chi_\nu + \chi_m + \chi_n) + \sum_{a, \beta} (\chi_a + \chi_\beta)$ ,  $\varepsilon_a = \varepsilon$ ,  $\varepsilon_\beta = -\varepsilon$ , the numbers of characters  $\chi_a$ ,  $\chi_\beta$  are  $\frac{1}{2}(s - 2 - \varepsilon)$ ,  $\frac{1}{2}(s - 2 + \varepsilon)$  respectively. Hence we have

$$x = 2\chi_\mu + \frac{1}{2}(s - 2 - \varepsilon) \chi_a + \frac{1}{2}(s - 2 + \varepsilon) \chi_\beta. \quad (B)$$

\* From the definition of  $x$  we get  $\sum \zeta_a = -\varepsilon$  or 0 according as  $\varepsilon_a = +\varepsilon$  or  $-\varepsilon$ . Hence we have in general  $\sum \zeta_a = -\varepsilon$ .

Eliminating  $\chi_\mu$  and  $\chi_\beta$  from (A) and (B) we obtain

$$sx = (s^2 + 1)\chi_\mu - f.$$

Substituting this value of  $sx$  in (10') we obtain

$$\chi_\mu(\chi_\mu - f) = 0.$$

If first  $\chi_\mu = 0$ , let  $f = s$ ; then  $x = y = -1$ ,  $\chi_\alpha = \epsilon$ ,  $\chi_\beta = -\epsilon$ , in general  $\chi_\epsilon = \epsilon_\epsilon$ , and therefore  $\chi(R^\alpha) = 1$ ,  $\chi(S^\beta) = -1$ . Also  $\chi_i = s$ .

If secondly  $\chi_\mu = f$ , let  $f = 1$ ; then  $\chi_i = \chi_\mu = \chi_m = \chi_\alpha = \chi_\beta = 1$ . Also  $x = s$ ,  $y = 0$ .

III. For all other solutions  $x = 0$ . Then (3) becomes  $\chi_\alpha \chi_\beta = 0$   $\epsilon_\beta = -\epsilon_\alpha$ . Not all the characters  $\chi_\epsilon$  can be zero. For, if they were, by giving to  $\epsilon_\alpha$  in (6) the values 1 and  $-1$  in turn we would have  $\chi_\mu + \chi_\nu = 0$ , and therefore  $f = 0$ , which is inadmissible.

According to (3) either all  $\chi_\alpha = 0$  for which  $\epsilon_\alpha = 1$ , or all for which  $\epsilon_\alpha = -1$ . Suppose first that  $\chi_\alpha = 0$  in case  $\epsilon_\alpha = -1$ ; and let  $\chi_i = f = s + 1$ . Then, since not all the characters  $\chi_\alpha$  for  $\epsilon_\alpha = 1$  can be zero, we obtain from (8)  $\chi_\mu = 1$ ; and from (12)  $\chi_m = 1$ . If  $\epsilon_\alpha = \epsilon_\beta = 1$ , and therefore  $\epsilon_\gamma = \epsilon_\delta = 1$ , we obtain from (4), (5), (6)

$$\chi_\alpha \chi_\beta = \chi_\gamma + \chi_\delta, \quad \chi_\alpha \chi_{-\alpha} = \chi_{-(2\alpha-1)} + 2, \quad \chi_\alpha^2 = \chi_{2\alpha-1} + 2.$$

If we set  $\chi_\alpha = \xi_a$ ,  $\xi_0 = \xi_{\frac{s-1}{2}} = 2$ , these equations can be combined into one:

$$\xi_a \xi_b = \xi_{a+b} + \xi_{a-b},$$

where  $a$  and  $b$  may be distinct or equal. Let  $r$  be a new unknown; if we set  $\xi_1 = r + r^{-1}$  it follows from  $\xi_1 \xi_1 = \xi_2 + \xi_0$  that  $\xi_2 = r^2 + r^{-2}$ ; then from  $\xi_1 \xi_2 = \xi_3 + \xi_1$  it follows that  $\xi_3 = r^3 + r^{-3}$ ; in general  $\xi_a = r^a + r^{-a}$ . From  $\xi_{\frac{s-1}{2}} = 2$  we get  $r^{\frac{s-1}{2}} = 1$ . We obtain then the solutions

$$\begin{aligned} \chi_i &= f = s + 1, \quad \chi_\mu = \chi_\nu = \chi_m = \chi_\alpha = 1, \\ \chi_\alpha &= r^a + r^{-a} \text{ if } \epsilon_\alpha = 1, \quad \chi_\beta = 0 \text{ if } \epsilon_\beta = -1. \end{aligned}$$

From (10''), (11'') we find  $y = z = -1$ . The above solutions satisfy the equation  $x = 0$  except when  $r = 1$ , and the equations  $y = z = -1$  except when  $r = -1$ ; and  $r$  can be  $-1$  only when  $\epsilon = 1$ . If  $\epsilon = 1$  the equation  $r^{\frac{s-1}{2}} = 1$  has  $\frac{s-5}{2}$  solutions distinct from  $\pm 1$ ; if  $\epsilon = -1$  it has  $\frac{s-3}{2}$  solutions distinct from 1; in general it has  $\frac{1}{2}(s-4-\epsilon)$  admissible solutions. Since  $r^a$  and  $r^{-a}$  give the same value for  $r^a + r^{-a}$  these solutions go in pairs, giving  $\frac{1}{2}(s-4-\epsilon)$  characters.

We next let  $\chi_i = -f = -(s+1)$ . Then  $\chi_m = -1$ ; and (4), (5), (6) become

$$\chi_a \chi_\beta = \chi_\gamma + \chi_\delta, \quad \chi_a \chi_{-a} = \chi_{-(2a^2-1)} - 2, \quad \chi_a^2 = \chi_{2a^2-1} + 2.$$

Setting  $\chi_a = \xi_a$ ,  $\xi_0 = 2$ ,  $\xi_{\frac{s-1}{2}} = -2$  we get  $\xi_a \xi_b = \xi_{a+b} + \xi_{a-b}$ . Let  $\xi_1 = r_1 + r_1^{-1}$ ; then as above we obtain  $\xi_a = r_1^a + r_1^{-a}$ , and also  $r_1^{\frac{s-1}{2}} = -1$ . We have the following solutions:

$$f = s+1, \quad \chi_i = -(s+1), \quad \chi_\mu = \chi_\nu = 1, \quad \chi_m = \chi_n = -1, \\ \chi_a = r_1^a + r_1^{-a} \text{ if } \varepsilon_a = 1, \quad \chi_\beta = 0 \text{ if } \varepsilon_\beta = -1.$$

Now  $x = 0$ ,  $y = z = \varepsilon - 1$ . The above solutions satisfy  $x = 0$ , and also  $y = z = \varepsilon - 1$  except when  $r_1 = -1$ , which can happen only when  $\varepsilon = -1$ . These solutions furnish  $\frac{1}{2}(s-2+\varepsilon)$  characters.

IV. Suppose finally that  $\chi_a = 0$  in case  $\varepsilon_a = 1$ . Let  $f = s-1$ . Assuming first that  $\chi_i = f$  we get  $\chi_m = \chi_\mu = -1$ . From (4), (5), (6) we have

$$\chi_a \chi_\beta = -\chi_\gamma - \chi_\delta, \quad \chi_a \chi_{-a} = -\chi_{-(2a^2-1)} + 2, \quad \chi_a^2 = -\chi_{2a^2-1} + 2.$$

Setting  $\chi_a = -\xi_a$ ,  $\xi_0 = \xi_{\frac{s+1}{2}} = 2$ , we obtain  $\xi_a \xi_b = \xi_{a+b} + \xi_{a-b}$ . If  $\xi_1 = t + t^{-1}$  then  $\xi_b = t^b + t^{-b}$ , and  $t^{\frac{s+1}{2}} = 1$ . We have then the following solutions:

$$\chi_i = f = s-1, \quad \chi_\mu = \chi_\nu = \chi_m = \chi_n = -1, \\ \chi_a = 0 \text{ if } \varepsilon_a = 1, \quad \chi_\beta = -(t^b + t^{-b}) \text{ if } \varepsilon_\beta = -1.$$

The equation  $x = 0$  is satisfied except when  $t = 1$ ; and the equations  $y = -1$ ,  $z = 1 - 2\varepsilon$  are satisfied except when  $t = -1$ , which can happen only when  $\varepsilon = -1$ . These solutions furnish  $\frac{1}{2}(s-2+\varepsilon)$  characters.

Assuming next that  $\chi_i = -f$  we get  $\chi_\mu = -1$ ,  $\chi_m = 1$ ; also

$$\chi_a \chi_\beta = -\chi_\gamma - \chi_\delta, \quad \chi_a \chi_{-a} = -\chi_{-(2a^2-1)} - 2, \quad \chi_a^2 = -\chi_{2a^2-1} + 2.$$

Setting  $\chi_a = -\xi_a$ ,  $\xi_0 = 2$ ,  $\xi_{\frac{s+1}{2}} = -2$ , we obtain  $\xi_a \xi_b = \xi_{a+b} + \xi_{a-b}$ . If  $\xi_1 = t_1 + t_1^{-1}$ , then  $\xi_b = t_1^b + t_1^{-b}$ , and  $t_1^{\frac{s+1}{2}} = 1$ . We have the following solutions:

$$f = s-1, \quad \chi_i = -(s-1), \quad \chi_\mu = \chi_\nu = -1, \quad \chi_m = \chi_n = 1, \\ \chi_a = 0 \text{ if } \varepsilon_a = 1, \quad \chi_\beta = -(t_1^b + t_1^{-b}) \text{ if } \varepsilon_\beta = -1.$$

We find that  $x = 0$  is satisfied by all these solutions; and that  $y = -(1+\varepsilon)$  and  $z = 1 + \varepsilon$  are satisfied by all except  $t_1 = -1$ , which can happen only when  $\varepsilon = 1$ . These solutions furnish  $\frac{1}{2}(s-\varepsilon)$  characters.

The total number of characters thus obtained is

$$4 + 2 + \frac{1}{2}(s-4-\varepsilon) + \frac{1}{2}(s-2+\varepsilon) + \frac{1}{2}(s-2+\varepsilon) + \frac{1}{2}(s-\varepsilon) = s+4.$$

which is equal to the number of classes of conjugate substitutions.

Finally, we readily find that for all these  $s+4$  characters the second proportionality factor  $e$  is equal to  $f$ , where  $e$  is defined by

$$\frac{hf}{e} = \sum_{\sigma} h_{\sigma} \chi_{\sigma} \chi_{\sigma'}$$

Below is given a table of the group-characters,  $N$  denoting the number of characters in the respective columns.

$N$	1	1	2	2	$\frac{s-\varepsilon-4}{4}$	$\frac{s+\varepsilon-2}{4}$	$\frac{s+\varepsilon-2}{4}$	$\frac{s-\varepsilon}{4}$
$\chi_{\lambda}$	1	$s$	$\frac{s+\varepsilon}{2}$	$\frac{s-\varepsilon}{2}$	$s+1$	$s+1$	$s-1$	$s-1$
$\chi_{\mu}$	1	$s$	$\frac{s+\varepsilon}{2}$	$-\frac{s-\varepsilon}{2}$	$s+1$	$-(s+1)$	$s-1$	$-(s-1)$
$\chi_{\nu}$	1	0	$\frac{\varepsilon \pm \sqrt{\varepsilon s}}{2}$	$-\frac{\varepsilon \pm \sqrt{\varepsilon s}}{2}$	1	1	-1	-1
$\chi_{\nu'}$	1	0	$\frac{\varepsilon \mp \sqrt{\varepsilon s}}{2}$	$-\frac{\varepsilon \mp \sqrt{\varepsilon s}}{2}$	1	1	-1	-1
$\chi_{\omega}$	1	0	$\frac{\varepsilon \pm \sqrt{\varepsilon s}}{2}$	$\frac{\varepsilon \mp \sqrt{\varepsilon s}}{2}$	1	-1	-1	1
$\chi_{\omega'}$	1	0	$\frac{\varepsilon \mp \sqrt{\varepsilon s}}{2}$	$\frac{\varepsilon \pm \sqrt{\varepsilon s}}{2}$	1	-1	-1	1
$\chi(R^a)$	1	1	$\frac{(1+\varepsilon)(-1)^a}{2}$	$\frac{(1-\varepsilon)(-1)^a}{2}$	$r^a + r^{-a}$	$r_1^a + r_1^{-a}$	0	0
$\chi(S^b)$	1	-1	$-\frac{(1-\varepsilon)(-1)^b}{2}$	$-\frac{(1+\varepsilon)(-1)^b}{2}$	0	0	$-t^b - t^{-b}$	$-t_1^b - t_1^{-b}$

where  $r, r_1, t, t_1$ , are the roots (except  $\pm 1$ ) of the respective equations  $r^{\frac{s-1}{2}} = 1$ ,  $r_1^{\frac{s-1}{2}} = -1$ ,  $t^{\frac{s+1}{2}} = 1$ ,  $t_1^{\frac{s+1}{2}} = -1$ .

## § 6.

By the use of the following theorems, due to Frobenius, we are able to deduce the group-characters of  $F$  from those of  $H$ .

*If  $G$  is an invariant subgroup of the group  $H$  then every character of  $\frac{H}{G}$  is also a character of  $H$ .\**

\* Über die Darstellung der endlichen Gruppen durch lineare Substitutionen, Berliner Sitzungsberichte, 1897, p. 995.

In order that a character of  $H$  may belong to the group  $\frac{H}{G}$  it is necessary and sufficient that it have the same value for all elements of  $G$ . Then it has also equal values for every two elements of  $H$  which are equivalent mod.  $G$ .\*

In the present case the invariant subgroup  $G$  of  $H$  is composed of the substitutions  $(l) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$  and  $(\lambda)$  the identity. Then those and only those characters of  $H$  for which  $\chi_l = \chi_\lambda$  belong to the group  $\frac{H}{G} = F$ ; and since every character of  $F$  belongs to  $H$ , we obtain in this way all the characters of  $F$ . The classes  $(\mu)$  and  $(m)$ , also  $(\nu)$  and  $(n)$ , are equivalent mod.  $G$ . Hence we can write down at once the table of characters for  $F$ .

$N$	1	1	2	$\frac{s - \varepsilon - 4}{4}$	$\frac{s + \varepsilon - 2}{4}$
$\chi_\lambda$	1	$s$	$\frac{s + \varepsilon}{2}$	$s + 1$	$s - 1$
$\chi_\mu$	1	0	$\frac{\varepsilon \pm \sqrt{\varepsilon s}}{2}$	1	-1
$\chi_\nu$	1	0	$\frac{\varepsilon \mp \sqrt{\varepsilon s}}{2}$	1	-1
$\chi(R^a)$	1	1	$\frac{(1 + \varepsilon)(-1)^a}{2}$	$r^a + r^{-a}$	0
$\chi(S^b)$	1	-1	$-\frac{(1 - \varepsilon)(-1)^b}{2}$	0	$-t^b - t^{-b}$

where  $r$  and  $t$  are the roots (except  $\pm 1$ ) of the respective equations  $r^{\frac{s-1}{2}} = 1$ ,  $t^{\frac{s+1}{2}} = 1$ .

## II.

*The Binary Linear Homogeneous Group  $H_1$  in the  $GF[2^n]$ .*

The order of  $H_1$  is  $h = 2^n(2^{2^n} - 1)$ , and the determinant of each substitution is unity. The group is holoedrically isomorphic with the group of all binary linear fractional substitutions in the  $GF[2^n]$ .

\*Über Relationen zwischen den Charakteren einer Gruppe und denen ihrer Untergruppen, Berliner Sitzungsberichte, 1898, p. 510.

We define  $\kappa = \alpha + \delta$  as the invariant of the substitution  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ .

The substitution  $R = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}$ , where  $\rho$  is a primitive root of the  $GF[2^n]$ , generates a cyclic group of order  $s - 1$ .  $R^a$  is conjugate to  $R^{-a}$  and is always distinct from it. Hence the powers of  $R$  represent  $\frac{s-2}{2}$  classes, each containing  $s(s+1)$  substitutions.

Let  $\sigma$  be a primitive root of  $\sigma^{s+1} = 1$ . The substitution  $S = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$  is of period  $s + 1$ .  $S^b$  is conjugate to  $S^{-b}$  and is distinct from it; thus the powers of  $S$  represent  $\frac{s}{2}$  classes, each containing  $s(s-1)$  substitutions.

The substitution  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  of period two and invariant zero is one of  $s^2 - 1$  conjugate substitutions. We denote this class by  $(0)$  and the identity by  $(\lambda)$ .

The total number of classes of conjugate substitutions is  $s + 1$ .

Below is given the table of group-characters.

$N$	1	1	$2^{n-1} - 1$	$2^{n-1}$
$\chi_\lambda$	1	$2^n$	$2^n + 1$	$2^n - 1$
$\chi_0$	1	0	1	-1
$\chi(R^a)$	1	1	$r^a + r^{-a}$	0
$\chi(S^b)$	1	-1	0	$-t^b - t^{-b}$

where  $r$  and  $t$  are the roots (except unity) of the respective equations  $r^{s-1} = 1$ ,  $t^{s+1} = 1$ . As before  $e = f$ .

### III.

*The Binary Linear Fractional Group  $F_1$  in the  $GF[p^n]$ ,  $p > 2$ , of all Determinants not Zero.*

The order of  $F_1$  is  $h = s(s^2 - 1)$ . The substitutions will be supposed written in the normal form, i. e., of determinant unity or a particular not-square in the  $GF[p^n]$ .

We shall denote the determinant  $\alpha\delta - \beta\gamma$  of the substitution  $V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  by  $\tau$ , where  $\tau = 1$ , or  $\nu$  a particular not-square; and we shall call  $\pm \frac{1}{2}(\alpha + \delta)$  the invariant of  $V$ .

If two substitutions have not the same determinant they are not conjugate. If two substitutions (neither the identity) have the same determinant and the same invariant, they are conjugate under  $F_1$ .

By canonical form theory we find that all the substitutions of the group can be reduced to one or another of the following canonical forms:

- A)  $R = \begin{pmatrix} \rho, & 0 \\ 0, & \rho^{-1}\tau \end{pmatrix}$ ,  $\rho$  a mark  $\neq 0$  of the  $GF[p^n]$ ;  
 B)  $S = \begin{pmatrix} \sigma, & 0 \\ 0, & \sigma^{-1}\tau \end{pmatrix}$ ,  $\sigma$  a mark  $\neq 0$  of the  $GF[p^n]$ ;  
 C)  $T = \begin{pmatrix} 1, & 1 \\ 0, & 1 \end{pmatrix}$ ,

where  $\sigma$  satisfies a quadratic equation belonging to and irreducible in the  $GF[p^n]$ .

A) The substitution

$$R = \begin{pmatrix} \rho, & 0 \\ 0, & \rho^{-1}\nu \end{pmatrix}$$

where  $\rho^2\nu^{-1}$  is a primitive root of the  $GF[p^n]$ , is of period  $s-1$ . With the exception of  $R^{\frac{s-1}{2}}$  which is conjugate only to itself,  $R^a$  is conjugate to  $R^{-a}$  and is distinct from it. We have therefore  $\frac{s-1}{2}$  classes represented by the powers of  $R$ , each containing  $s(s+1)$  substitutions, except  $R^{\frac{s-1}{2}}$ , the class represented by which contains  $\frac{1}{2}s(s+1)$  substitutions.

B) The group of all binary linear fractional substitutions in the  $GF[p^n]$  of determinant  $\neq 0$  is holodrically isomorphic with the group\* of binary hyperorthogonal substitutions in the  $GF[p^n]$  of determinant a mark of the  $GF[p^n]$  when taken fractionally, viz.,

$$U = \begin{pmatrix} A, & B \\ -\bar{B}, & \bar{A} \end{pmatrix} \quad (A\bar{A} + B\bar{B} = \pi),$$

where  $\bar{A} \equiv A^*$  is the conjugate of  $A$  with respect to the  $GF[p^n]$ , and  $\pi$  is a mark  $\neq 0$  of the  $GF[p^n]$ .

Consider the substitution

$$S = \begin{pmatrix} \sigma, & 0 \\ 0, & \sigma^{-1}\nu \end{pmatrix},$$

where  $\sigma$  is a primitive root of the equation  $\sigma^{s+1} = \nu$ . Since  $\nu$  is an *arbitrary* not-square we may suppose that it is a primitive root of the  $GF[p^n]$ . Then  $\sigma$  is a

\* Dickson, *Linear Groups*, § 144, Cor.

primitive root of the  $GF[p^{2n}]$ , and consequently  $S$  is of period  $s + 1$ . With the exception of  $S^{\frac{s+1}{2}}$  which is conjugate only to itself,  $S^b$  is conjugate to  $S^{-b}$  and is distinct from it. We have therefore  $\frac{s+1}{2}$  classes represented by the powers of  $S$ , each containing  $s(s-1)$  substitutions, except  $S^{\frac{s+1}{2}}$ , the class represented by which contains  $\frac{1}{2}s(s-1)$  substitutions.

The classes represented by the powers of  $R(S)$  are characterized by the property that  $x^2 - \tau$  is a square (not-square) in the  $GF[p^n]$ , where  $\tau = v$  or 1 according as the index is odd or even.

The substitution

$$T_\mu = \begin{pmatrix} 1, \mu \\ 0, 1 \end{pmatrix}, \mu \text{ a mark } \neq 0 \text{ of the } GF[p^n],$$

of invariant  $\pm 1$  and determinant unity, is one of  $s^2 - 1$  conjugate substitutions forming a class ( $\mu$ ).

The total number of classes of conjugate substitutions is  $s + 2$ .

Below is given the table of group-characters.

$N$	1	1	1	1	$\frac{s-3}{2}$	$\frac{s-1}{2}$
$\chi_\lambda$	1	1	$s$	$s$	$s+1$	$s-1$
$\chi_\mu$	1	1	0	0	1	-1
$\chi(R^{2a})$	1	1	1	1	$r^{2a} + r^{-2a}$	0
$\chi(S^{2b})$	1	1	-1	-1	0	$-t^{2b} - t^{-2b}$
$\chi(R^{2a+1})$	1	-1	1	-1	$r^{2a+1} + r^{-(2a+1)}$	0
$\chi(S^{2b+1})$	1	-1	-1	1	0	$-t^{2b+1} - t^{-(2b+1)}$

where  $r$  and  $t$  are the roots (except  $\pm 1$ ) of  $r^{s-1} = 1$  and  $t^{s+1} = 1$  respectively. As before  $e = f$ .

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### VITA.

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